

CONNECTEDNESS OF BRILL-NOETHER LOCI VIA DEGENERATIONS

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ABSTRACT. We show that limit linear series spaces for chains of curves are reduced. Using new advances in the foundations of limit linear series, we then use degenerations to study the question of connectedness for spaces of linear series with imposed ramification at up to two points. We find that in general, these spaces may not be connected even when they have positive dimension, but we prove a criterion for connectedness which generalizes the theorem previously proved by Fulton and Lazarsfeld in the case without imposed ramification.

1. INTRODUCTION

The classical Brill-Noether theorem states that if we are given $g, r, d \geq 0$, a general curve X of genus g carries a linear series (\mathcal{L}, V) of rank r and degree d if and only if the quantity

$$\rho := g - (r + 1)(r + g - d)$$

is nonnegative [GH80]. Moreover, in this case the moduli space $G_d^r(X)$ of such linear series has pure dimension ρ . This was generalized by Eisenbud and Harris to allow for imposed ramification: if we have marked points $P_1, \dots, P_n \in X$, and sequences $0 < a_0^i < \dots < a_r^i \leq d$ for $i = 1, \dots, n$, we can consider the moduli space $G_d^r(X, (P_1, a_\bullet^1), \dots, (P_n, a_\bullet^n)) \subseteq G_d^r(X)$ parametrizing linear series with vanishing sequence at least a_\bullet^i at each of the P_i . Then Eisenbud and Harris used their theory of limit linear series to show that in characteristic 0, if (X, P_1, \dots, P_n) is a general n -marked curve of genus g , the dimension of $G_d^r(X, (P_1, a_\bullet^1), \dots, (P_n, a_\bullet^n))$ – if it is nonempty – is given by the generalized formula

$$\rho := g - (r + 1)(r + g - d) - \sum_{i=1}^n \sum_{j=0}^r (a_j^i - j).$$

The condition for nonemptiness is still numerical, but becomes more complicated in this context. This theorem fails in positive characteristic for $n \geq 3$, but is still true if $n \leq 2$; in this case, we also have a simple criterion for nonemptiness. See for instance [Oss14b] for a proof of the following.

Theorem 1.1. *Given (g, r, d) nonnegative integers, and sequences $0 \leq a_0 < a_1 < \dots < a_r \leq d$, $0 \leq b_0 < b_1 < \dots < b_r \leq d$, set*

$$(1.1) \quad \hat{\rho} := g - \sum_{j: a_j + b_{r-j} > d-g} a_j + b_{r-j} - (d - g).$$

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Then, if X is a general (smooth, projective) curve of genus g and $P, Q \in X$ are general points, the moduli space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is nonempty if and only if $\hat{\rho} \geq 0$, and in this case, it has pure dimension ρ .

In a complementary direction, Fulton and Lazarsfeld used an analysis of degeneracy loci on $\text{Pic}^d(X)$ to show that the spaces $G_d^r(X)$ are always connected when $\rho \geq 1$ [FL81]. Our main result is the following theorem combining these two strands:

Theorem 1.2. *In the situation of Theorem 1.1, the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is also reduced whenever it is nonempty.*

Furthermore, if (X, P, Q) is any 2-marked curve of genus g such that the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is pure of dimension ρ , and if $\hat{\rho} \geq 1$, then $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is connected.

Note that in the situation of the theorems, ρ is given by the same formula as (1.1), but with the sum ranging over all j . Thus, $\hat{\rho} \leq \rho$ always. We will have $\hat{\rho} = \rho$ whenever $d \leq r + g$, which underlines that for questions involving imposed ramification, the nonspecial case often displays interesting and novel behavior. On the other hand, when no ramification is imposed, we will have $\hat{\rho} = g$ whenever $d > r + g$, so we recover the Fulton-Lazarsfeld result on connectedness whenever $\rho \geq 1$. Example 4.4 demonstrates that in the presence of imposed ramification, the hypothesis that $\hat{\rho} \geq 1$ is indeed necessary for connectedness. Together with the criterion for nonemptiness stated in Theorem 1.1, a pattern emerges that in generalizing from classical statements without imposed ramification, the quantity $\hat{\rho}$ seems to arise naturally as an alternative generalization of ρ .

Aside from the generalization to imposed ramification, the novelty of our approach is that until now, the connectedness theorem was the only part of standard Brill-Noether theory which did not have a proof using degeneration techniques. The main reason for this was not that the analysis of the relevant limit linear series spaces was especially difficult, but that to study topological properties such as connectedness via specializations, it is crucial to have flat, proper families with well-understood scheme structures. Until the machinery introduced in [Oss14a] and [MO16], such moduli spaces had never been constructed in the context of limit linear series. It is also a pleasant facet of our techniques that they are intrinsically characteristic independent. Finally, we mention that even when $\hat{\rho} = 0$, our techniques reduce describing the number of connected components of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ to a purely combinatorial problem; see Remark 4.6.

Note that our connectedness theorem does not strictly generalize the Fulton-Lazarsfeld theorem because it applies to Brill-Noether-general curves, whereas their result applies to all curves. While reducedness was known in the absence of imposed ramification going back to Griffiths and Harris [GH80], and is in any case superseded by the Gieseker-Petri theorem, it appears to be new in the case of imposed ramification, except that the case $\rho \leq 1$ is proved by Chan, Lopez, Pflueger, and Teixidor i Bigas in [CLPT]. The case of ramification at three or more points is much more delicate; see Remark 3.6.

In the case that $\rho = 0$ and there is no imposed ramification, spaces of linear series are not connected, but (in characteristic 0) Eisenbud and Harris proved [EH87] that in suitable families, the relative space of linear series is still connected. This leads us to ask:

Question 1.3. In cases with $\widehat{\rho} = 0$, when the space of linear series with imposed ramification on individual curves is not connected, are there families of curves over which the relative space of linear series is nonetheless connected?

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Conventions. We work throughout over an algebraically closed field of arbitrary characteristic.

Given $r, d \geq 0$, a **vanishing sequence** $a_\bullet = a_0, \dots, a_r$ is a strictly increasing sequence with $a_0 \geq 0$ and $a_r \leq d$. Two vanishing sequences a_\bullet, b_\bullet are **complementary** if $a_j + b_{r-j} = d$ for $j = 0, \dots, r$. Given vanishing sequences a_\bullet, a'_\bullet , we write $a_\bullet \geq a'_\bullet$ if $a_j \geq a'_j$ for $j = 0, \dots, r$, and we write $a_\bullet > a'_\bullet$ if $a_\bullet \geq a'_\bullet$ and furthermore $a_j > a'_j$ for some j .

2. RICHARDSON VARIETIES AND THE BASE CASES

We recall that a **Richardson variety** is an intersection of two Schubert varieties associated to transverse flags. These are well studied, starting with Richardson [Ric92]: they are irreducible and reduced of the expected codimension; it follows from the Cohen-Macaulayness of Schubert varieties that they are also Cohen-Macaulay. Richardson varieties arise naturally in studying the fibers of the map

$$G_d^r(X, (P, a_\bullet), (Q, b_\bullet)) \rightarrow \text{Pic}^d(X);$$

in fact, in our base cases where X has genus 0 or 1, the general fibers will always be Richardson varieties. This is the basis for the proof of the following.

Proposition 2.1. *Given r, d nonnegative integers, and sequences $0 \leq a_0 < a_1 < \dots < a_r \leq d$, $0 \leq b_0 < b_1 < \dots < b_r \leq d$, let X be a smooth, projective curve of genus g and $P, Q \in X$, and suppose either that $g = 0$ and $P \neq Q$, or that $g = 1$ and $P - Q$ is not m -torsion for any $m \leq d$. Then if $\widehat{\rho} \geq 0$, the moduli space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is reduced and irreducible, and Cohen-Macaulay of dimension ρ .*

It is already shown for instance in Lemma 2.1 of [Oss14b] that under the hypotheses of Proposition 2.1, the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is nonempty if and only if $\widehat{\rho}$ is satisfied, and in this case it has pure dimension ρ . As indicated above, the basic idea is to describe most of the fibers of the morphism $G_d^r(X, (P, a_\bullet), (Q, b_\bullet)) \rightarrow \text{Pic}^d(X)$ as Richardson varieties, and a more careful analysis of this argument will yield a proof of Proposition 2.1. In fact, there was an oversight in one case of the argument in [Oss14b], which we will address simultaneously.

Proof. First, if X has genus 0, the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is identified with an intersection of two Schubert varieties in the Grassmannian $G(r+1, \Gamma(X, \mathcal{O}(d))) \cong G(r+1, d+1)$. Moreover, one checks easily that the flags defining these Schubert varieties meet transversely, so in this case $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is itself a Richardson variety, and the desired statement follows immediately.

On the other hand, if X has genus 1, we elaborate on the argument given in Lemma 2.1 of [Oss14b]. As mentioned above, we have nonemptiness in this case if and only if $a_j + b_{r-j} \leq d$, with equality occurring for at most one j . If equality does occur for some j_0 , then $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is supported over the point of

$\text{Pic}^d(X)$ corresponding to $\mathcal{O}_X(a_{j_0}P + b_{r-j_0}Q) = \mathcal{O}_X(a_{j_0}P + (d - a_{j_0})Q)$, whereas if the inequality is strict for all j , it maps surjectively onto $\text{Pic}^d(X)$. We refer to these as the first and second cases, respectively.

Our first task is to verify that in the first case, $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is supported scheme-theoretically over the relevant point of $\text{Pic}^d(X)$. Accordingly, let T be a k -scheme, and $(\mathcal{L}, \mathcal{V})$ a T -valued point of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$, so that in particular \mathcal{L} is a line bundle on $X \times_k T$ and \mathcal{V} is a rank- $(r+1)$ vector bundle on T with a map to $p_{2*}\mathcal{L}$. For brevity, write $a = a_{j_0}$. Then by definition of imposed ramification, we have that the rank of the natural maps $\mathcal{V} \rightarrow p_{2*}(\mathcal{L}|_{aP \times T})$ and $\mathcal{V} \rightarrow p_{2*}(\mathcal{L}|_{(d-a)Q \times T})$ are (scheme-theoretically) less than or equal to j_0 and $r - j_0$ respectively on T . We then see by expanding out minors in local matrix expressions that the natural map

$$(2.1) \quad \mathcal{V} \rightarrow p_{2*}(\mathcal{L}|_{aP \times T}) \oplus p_{2*}(\mathcal{L}|_{(d-a)Q \times T}) = p_{2*}(\mathcal{L}|_{(aP + (d-a)Q) \times T})$$

has rank less than or equal to r on T . On the other hand, at any point $t \in T$, the map $\mathcal{V}|_t \rightarrow \Gamma(X, \mathcal{L}|_{(aP + (d-a)Q) \times t})$ can have only a 1-dimensional kernel, so we conclude that (2.1) has rank exactly r on T , and therefore contains a rank-1 subbundle \mathcal{M} which gives the kernel universally. Pulling back to $X \times_k T$, we have maps

$$p_{2*}\mathcal{M} \rightarrow \mathcal{V} \otimes \mathcal{O}_{X \times_k T} \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_{(aP + (d-a)Q) \times T}$$

such that the composed map to \mathcal{L} does not vanish identically in any fiber, but the composed map to $\mathcal{L}|_{(aP + (d-a)Q) \times T}$ is zero. The latter implies that the map factors through a map $p_{2*}\mathcal{M} \rightarrow \mathcal{L}(-(aP + (d-a)Q) \times T)$ which does not vanish identically in any fiber. Since both line bundles have relative degree 0, we conclude that this map is an isomorphism, and hence that our T -valued point is supported (scheme-theoretically) in the desired fiber of $\text{Pic}^d(X)$.

Next, in both the first and second cases, if $\mathcal{L} \in \text{Pic}^d(X)$ is in the image of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$, then the fiber over \mathcal{L} is again described (scheme-theoretically) as an intersection of two Schubert varieties inside a Grassmannian $G(r+1, \Gamma(X, \mathcal{L})) \cong G(r+1, d)$. We recall from [Oss14b] that when \mathcal{L} is not of the form $\mathcal{O}(aP + (d-a)Q)$ for some a with $0 \leq a \leq d$, then the intersection is again a Richardson variety, of dimension $\rho - 1$. If $\mathcal{L} = \mathcal{O}(dP)$, then we still obtain a Richardson variety which will have dimension $\rho - 1$ if $a_r < d$ and ρ if $a_r = d$ (due to the difference in this case between the vanishing sequence indexing and the usual Schubert variety indexing). The same holds if $\mathcal{L} = \mathcal{O}(dQ)$, with b_r in place of a_r . The final situation is that $\mathcal{L} = \mathcal{O}(aP + (d-a)Q)$ for some a with $0 < a < d$. Here, our flags do not intersect transversely, at precisely the place corresponding to imposing order- a vanishing at P and order- $(d-a)$ vanishing at Q . If either a does not appear in a_\bullet or $d-a$ does not occur in b_\bullet , the partial flags used to define the Schubert varieties are still transverse, so the intersection is still a Richardson variety, of dimension $\rho - 1$. Next, we consider the possibility that $a = a_{j_0}$ and $d-a = b_{r-j_0}$ for some j_0 (i.e., exactly the situation occurring in our first case). As explained in [Oss14b], in this case we do not change the intersection of Schubert varieties if we replace a_{j_0} by $a - 1$, and we then see that we again get a Richardson variety, but this time of dimension ρ . *A priori* this argument was set-theoretic, but since replacing a by $a - 1$ in a_\bullet only increases the size of the Schubert variety, we have that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is a closed subscheme of the Richardson variety, with the same support, and since the latter is reduced they must agree as schemes.

This completes the argument for the first case, and for the second case, the last possibility we have to consider above is that we have $a = a_{j_1}$ and $d - a = b_{r-j_2}$ for some $j_1 \neq j_2$. This case was overlooked in [Oss14b], and in fact in this case the fibers may be reducible, but in any case, the dimension of the fiber over $\text{Pic}^d(X)$ is at most $\rho - 1$ in this situation. The argument can be expressed in terms of rather standard descriptions of intersections of pairs of Schubert varieties associated to non-transverse flags (as described e.g. in §§2.2 and 2.5 and Lemma 2.6 of [Vak06]), and in this context turns out to be a special case of the last paragraph of the proof of Lemma 6.3 of [Oss16]. However, because the bound can also be obtained by a brief direct argument, for the sake of remaining self contained we include a proof. First, any linear series (\mathcal{L}, V) in $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ has some vanishing sequences $a'_\bullet \geq a_\bullet$ and $b'_\bullet \geq b_\bullet$ at P and Q , and it is straightforward to see that V must admit a basis s_0, \dots, s_r such that $\text{ord}_P s_j = a'_j$ and $\text{ord}_Q s_j = b'_{\sigma(r-j)}$ for some permutation σ of $0, \dots, r$. The data of a'_\bullet, b'_\bullet and σ is equivalent to specifying the dimension of incidence of V with every simultaneous vanishing condition at P and Q , so if we fix $\mathcal{L} = \mathcal{O}(aP + (d-a)Q)$, we obtain a decomposition of the fiber of interest into locally closed subsets, each of which we want to show has dimension at most $\rho - 1$. But for each subset, we have a surjection from the space of $(r+1)$ -tuples of sections s_j of \mathcal{L} satisfying the above conditions on orders at P and Q . This space of tuples has dimension

$$\sum_{j=0}^r \dim \Gamma(X, \mathcal{L}(-a'_j P - b'_{\sigma(r-j)} Q)),$$

and each term in the sum is equal to $d - a'_j - b'_{\sigma(r-j)}$ except in the case that $a'_j = a$ and $b'_{\sigma(r-j)} = d - a$, in which case we get $d - a'_j - b'_{\sigma(r-j)} + 1$. Thus, the sum is equal to $(r+1)d - \sum_j a'_j - \sum_j b'_j + \epsilon$, where $\epsilon = 1$ if there is some j with $a'_j = a$ and $b'_{\sigma(r-j)} = d - a$, and $\epsilon = 0$ otherwise. The hypotheses of our situation entail that if $\epsilon = 1$, then $\sigma(r-j) \neq r-j$, so in particular $\sigma \neq \text{id}$. Now, the fibers from the space of $(r+1)$ -tuples to $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ always have dimension at least $r+1$, corresponding to independent scaling of the s_j , and we see that additional changes of basis are possible (and hence the fiber dimension is strictly bigger) precisely when $\sigma \neq \text{id}$, since in this case there is necessarily some $j_1 < j_2$ with $\sigma(r-j_1) < \sigma(r-j_2)$. We thus compute that the dimension of each given subset of the fiber is bounded by

$$(r+1)d - \sum_j a'_j - \sum_j b'_j - (r+1) \leq (r+1)d - \sum_j a_j - \sum_j b_j - (r+1) = \rho - 1,$$

as desired.

In the second case, we now have that the map $G_d^r(X, (P, a_\bullet), (Q, b_\bullet)) \rightarrow \text{Pic}^d(X)$ is proper and surjective, with every fiber having dimension at most $\rho - 1$, and with the general fiber reduced and irreducible. We also know *a priori* that every component of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ has dimension at least ρ , so we conclude that every component must have dimension exactly ρ , every fiber must have dimension $\rho - 1$, and no component can be supported in any fiber. It then follows from the irreducibility of $\text{Pic}^d(X)$ that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is irreducible. Now, in the second case we cannot have $a_r = d$ or $b_r = d$, so we see that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is constructed inside the relative Grassmannian $G_d^r(X)$ as an intersection of two relative Schubert varieties, which are each Cohen-Macaulay. Since we have shown

that this intersection has the expected codimension, it is Cohen-Macaulay. This implies first that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is flat over $\text{Pic}^d(X)$. But then we conclude that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is generically reduced, since it is irreducible and the general fiber is reduced. Finally, reducedness follows as an additional consequence of Cohen-Macaulayness. \square

3. LIMIT LINEAR SERIES

We now use the Eisenbud-Harris theory of limit linear series to prove the reducedness portion of Theorem 1.2. For the sake of convenience, we also incorporate the well-known dimension results into our statements. We begin by specifying the curves we will consider, and recalling the basic definitions.

Situation 3.1. Fix g, d, n . Let Z_1, \dots, Z_n be smooth projective curves, with (distinct) P_i, Q_i on Z_i for each i , and let X_0 be the nodal curve obtained by gluing Q_i to P_{i+1} for $i = 1, \dots, n-1$.

Definition 3.2. Given r, d a **limit linear series** of rank r and degree d on X_0 consists of a tuple (\mathcal{L}^i, V^i) of linear series of rank r and degree d on the Z_i , satisfying the following condition: if a_\bullet^i, b_\bullet^i are the vanishing sequences of (\mathcal{L}^i, V^i) at P_i and Q_i respectively, then we require

$$(3.1) \quad b_j^i + a_{r-j}^{i+1} \geq d$$

for all $i = 1, \dots, n-1$ and $j = 0, \dots, r$. If (3.1) is an equality for all i, j , we say that the limit linear series is **refined**.

The space of all such limit linear series on X_0 is denoted by $G_d^r(X_0)$. If we have sequences a_\bullet and b_\bullet , we also have the closed subscheme $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet)) \subseteq G_d^r(X_0)$ consisting of limit linear series such that, following the above notation, we have $a_\bullet^1 \geq a_\bullet$ and $b_\bullet^n \geq b_\bullet$.

Any set of vanishing sequences at all of the P_i, Q_i satisfying (3.1) can be decreased to obtain a set of refined vanishing sequences, so by definition we have that $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ is the union over all tuples of complementary vanishing sequences $b_\bullet^i, a_\bullet^{i+1}$ of

$$G_d^r(Z_1, (P_1, a_\bullet), (Q_i, b_\bullet^i)) \times \left(\prod_{i=2}^{n-1} G_d^r(Z_i, (P_i, a_\bullet^i), (Q_i, b_\bullet^i)) \right) \times G_d^r(Z_n, (P_i, a_\bullet^i), (Q_i, b_\bullet)).$$

We define the scheme structure on $G_d^r(X_0, (P_1, a_\bullet^1), (Q_n, b_\bullet^n))$ to be the one induced by the natural scheme structures on the individual spaces $G_d^r(Z_i, (P_i, a_\bullet^i), (Q_i, b_\bullet^i))$. To settle questions of reducedness and dimension it therefore suffices to consider the individual spaces $G_d^r(Z_i, (P_i, a_\bullet^i), (Q_i, b_\bullet^i))$. Note also that by additivity of the Brill-Noether number (Lemma 3.6 of [EH86]), expected dimension of the latter spaces implies that the limit linear series space has the expected dimension as well. In particular, when every Z_i has genus at most 1, Proposition 2.1 has the following immediate corollary.

Corollary 3.3. *Suppose that the genus of each Z_i is at most 1, and that for all i with Z_i having genus 1, we have that $P_i - Q_i$ is not m -torsion for any $m \leq d$. Given vanishing sequences a_\bullet, b_\bullet , the space $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ of limit linear series on X_0 is reduced of pure dimension ρ .*

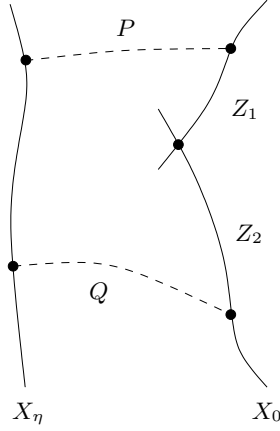


FIGURE 1. The degeneration considered in Theorem 3.4 in the case $n = 2$.

We next use the machinery of [Oss14a] and [MO16] to generalize to the case of components of higher genus. We first state the following easy consequence of [MO16].

Theorem 3.4. *Let B be the spectrum of a discrete valuation ring, and $\pi : X \rightarrow B$ be a flat, proper family of curves of genus g , with X regular, the generic fiber X_η smooth, and the special fiber isomorphic to X_0 . Further assume that π has sections P, Q , specializing to P_1 and Q_n respectively on X_0 .*

Suppose that for all vanishing sequences a_\bullet and b_\bullet with $\hat{\rho} \geq 0$, the spaces $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ are reduced of expected dimension ρ , respectively. Then for each a_\bullet, b_\bullet , there is a moduli space $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ proper and flat over B , with special and generic fibers isomorphic to $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ and $G_d^r(X_\eta, (P_\eta, a_\bullet), (Q_\eta, b_\bullet))$, respectively.

Proof. Set $\rho_0 = g - (r + 1)(g + r - d)$. The hypotheses imply that $G_d^r(X_0)$ is reduced of dimension ρ_0 , with the refined limit linear series forming a dense open subset, since any additional ramification results in dimension strictly smaller than ρ_0 . Then according to Corollary 3.4 of [MO16], there is a scheme $G_d^r(X/B)$ which is flat and proper over B , with generic fiber isomorphic to $G_d^r(X_\eta)$, and with special fiber isomorphic to $G_d^r(X_0)$. If we now impose vanishing sequence at least a_\bullet along P and at least b_\bullet along Q , we obtain a scheme $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ still proper over B , with generic fiber isomorphic to $G_d^r(X_\eta, (P_\eta, a_\bullet), (Q_\eta, b_\bullet))$, and special fiber isomorphic to $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$. By hypothesis, the latter is reduced, of pure dimension ρ . Moreover, the construction of $G_d^r(X/B)$ inside of an ambient space smooth over B ensures that it has universal relative dimension at least ρ_0 in the sense of Definition 3.1 of [Oss15], and considering the imposition of the additional ramification conditions inside of the same smooth ambient space we see that $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ has universal relative dimension at least ρ . It then follows from the reducedness of $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ and Proposition 3.7 of [Oss15] that $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ is flat over B , as desired. \square

We now conclude the following general reducedness statement. Note that the reducedness portion of Theorem 1.2 is the special case with $n = 1$.

Corollary 3.5. *Suppose that the components of X_0 are general as 2-marked curves. Given vanishing sequences a_\bullet, b_\bullet , the space $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ of limit linear series on X_0 is reduced of pure dimension ρ .*

Proof. We induct on genus, assuming that we know the desired statement for irreducible curves of genus strictly less than g , and proving it for irreducible curves of genus g . As discussed above, this implies the desired statement also for spaces of limit linear series when all components have genus at most g . The base case is that g is 0 or 1, so is handled by Proposition 2.1. Consider $n = 2$, with Z_1 a smooth projective curve of genus 1, and $P_1, Q_1 \in Z_1$ marked points not differing by m -torsion for any $m \leq d$, and with Z_2 a general smooth projective curve of genus $g - 1$, and $P_2, Q_2 \in Z_2$ general marked points. Then let $\pi : X \rightarrow B$ be as in Theorem 3.4. By the induction hypothesis, $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ is reduced of pure dimension ρ for all choices of a_\bullet, b_\bullet , so Theorem 3.4 gives us that $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ is flat and proper over B , and then Theorem 12.2.1 of [GD66] implies that the generic fiber $G_d^r(X_\eta, (P_\eta, a_\bullet), (Q_\eta, b_\bullet))$ is geometrically reduced. Since this condition is open in families, we conclude that the corollary holds for irreducible curves of genus g , as desired. \square

Remark 3.6. When we consider ramification at three or more points, the situation is much more delicate, even in genus 0. For instance, in positive characteristic, spaces of limit linear series frequently have greater than the expected dimension due to inseparability phenomena. Even over the complex numbers, reducedness of spaces of linear series with prescribed ramification is quite difficult: the case of genus 0 and $\rho = 0$ was proved relatively recently by Mukhin, Tarasov and Varchenko [MTV09] while proving the sharper Shapiro-Shapiro conjecture, but they used a connection to quantum algebra, and there is still no direct geometric proof. The case of higher genus then follows via limit linear series techniques; see [Oss03].

4. CONNECTEDNESS

Having established the necessary reducedness results, we can now prove the connectedness portion of Theorem 1.2.

We have the following preliminary lemma.

Lemma 4.1. *Suppose $\pi : Y \rightarrow B$ is flat and proper, with geometrically reduced fibers, and B is locally Noetherian and connected. Let $Z \subseteq Y$ be a closed subscheme which is also flat over B . If there exists $b \in B$ such that every connected component of Y_b meets Z , then the same is true for all $b \in B$.*

Proof. As B is assumed connected, it is enough to show that the set of $b \in B$ such that every connected component of Y_b meets Z is closed under both specialization and generization. Observe that the condition on Y_b is invariant under extension of base field, because the image of a connected component under a base field extension is a connected component (see for instance Tag 04PZ of [Sta17], although the proof is easier in the finite type case). Thus, the statement reduces to the case that B is the spectrum of a DVR. In fact, following the reasoning in the proof of Proposition 15.5.7 of [GD66], we may further assume that the connected components of the fibers are geometrically connected, and we then obtain that Y decomposes as a disjoint union of connected components, each with a unique connected component in both fibers. In this situation, the desired statement follows immediately from the flatness and properness of Z . \square

We also have the following lemma on subadditivity of $\widehat{\rho}$, which follows easily from the definitions.

Lemma 4.2. *Given $g, r, d, a_\bullet, b_\bullet$ as in Theorem 1.1, suppose we have also b_\bullet^1 and a_\bullet^2 complementary vanishing sequences, and g_1, g_2 nonnegative with $g_1 + g_2 = g$. Define $\widehat{\rho}_1$ using g_1 and b_\bullet^1 in place of g and b_\bullet , and define $\widehat{\rho}_2$ using g_2 and a_\bullet^2 in place of g and a_\bullet . Then we have*

$$\widehat{\rho} \geq \widehat{\rho}_1 + \widehat{\rho}_2,$$

with equality occurring if and only if for every j with $a_j + b_{r-j}^1 > d - g_1$, we have $a_j^2 + b_{r-j} \geq d - g_2$, and for every j with $a_j^2 + b_{r-j} > d - g_2$, we have $a_j + b_{r-j}^1 \geq d - g_1$.

In order to prove the connectedness portion of Theorem 1.2, we will need to induct on a more refined statement that involves also the case $\widehat{\rho} = 0$. Specifically, we prove the following:

Theorem 4.3. *In the situation of Theorem 1.1, if we have $\widehat{\rho} \geq 1$, then the moduli space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is connected.*

If $\widehat{\rho} = 0$, and if for some j_0 we can define a vanishing sequence \bar{a}_\bullet by replacing a_{j_0} with $a_{j_0} + 1$, while maintaining the condition $\widehat{\rho} = 0$, then every connected component of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ has nonempty intersection with the closed subscheme $G_d^r(X, (P, \bar{a}_\bullet), (Q, b_\bullet))$.

Note that the hypotheses on \bar{a}_\bullet in the second part of the theorem imply in particular that $\rho \geq 1$.

Proof. We begin by proving the second statement by induction on g . The base case that $g \leq 1$ is satisfied trivially because in this case $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is always connected. For the induction step, we again consider a family $\pi : X \rightarrow B$ just as in the proof of Corollary 3.5. We now know that $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ is reduced and that $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ is flat and proper over B , so according to Proposition 15.5.7 of [GD66], the number of (geometrically) connected components of $G_d^r(X/B, (P, a_\bullet), (Q, b_\bullet))$ is the same in the special and generic fibers. The same statements also hold for $G_d^r(X/B, (P, \bar{a}_\bullet), (Q, b_\bullet))$. Our main task is to show that every connected component of the limit linear series space $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ has nonempty intersection with the closed subscheme $G_d^r(X_0, (P_1, \bar{a}_\bullet), (Q_2, b_\bullet))$. We will analyze the situation on each part

$G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))^{b_\bullet^1, a_\bullet^2} := G_d^r(Z_1, (P_1, a_\bullet), (Q_1, b_\bullet^1)) \times G_d^r(Z_2, (P_2, a_\bullet^2), (Q_2, b_\bullet))$ of $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$, as b_\bullet^1, a_\bullet^2 range over complementary vanishing sequences with $\widehat{\rho}_1$ and $\widehat{\rho}_2$ both nonnegative.

First, if $a_{j_0} + b_{r-j_0}^1 < d - 1$, we see that the closed subset

$$G_d^r(Z_1, (P_1, \bar{a}_\bullet), (Q_1, b_\bullet^1)) \times G_d^r(Z_2, (P_2, a_\bullet^2), (Q_2, b_\bullet))$$

will still be nonempty, and because $G_d^r(Z_1, (P_1, a_\bullet), (Q_1, b_\bullet^1))$ is connected, we immediately obtain the desired statement, so it is enough to consider the cases that $a_{j_0} + b_{r-j_0}^1 = d$ or $a_{j_0} + b_{r-j_0}^1 = d - 1$. On the other hand, our hypothesis that $\widehat{\rho} \geq 0$ is maintained under replacing a_\bullet with \bar{a}_\bullet implies that $a_{j_0} + b_{r-j_0} < d - g$, and then by Lemma 4.2 we see that we cannot have $a_{j_0} + b_{r-j_0}^1 = d$. Thus, it remains to consider the case that $a_{j_0} + b_{r-j_0}^1 = d - 1$. In this case, we necessarily have $a_{j_0}^2 + b_{r-j_0} < d - (g - 1)$, and we claim that if $j_0 < r$, we must have $a_{j_0+1}^2 > a_{j_0}^2 + 1$:

indeed, we must have $a_{j_0+1} > a_{j_0} + 1$ by the validity of \bar{a}_\bullet , so the only way we could have $a_{j_0+1}^2 = a_{j_0}^2 + 1$ would be if $a_{j_0+1} + b_{r-j_0-1}^1 = d$. But then because $\hat{\rho} = 0$, Lemma 4.2 would imply that $a_{j_0+1}^2 + b_{r-j_0-1} \geq d - (g - 1)$, yielding the contradiction that $b_{r-j_0} \leq b_{r-j_0-1}$. Thus, we can obtain \bar{b}_\bullet^1 and \bar{a}_\bullet^2 by replacing $b_{r-j_0}^1$ with $b_{r-j_0}^1 - 1$ and $a_{j_0}^2$ with $a_{j_0}^2 + 1$ respectively, and this maintains the condition $\hat{\rho}_2 \geq 0$. The induction hypothesis implies that every connected component of $G_d^r(Z_2, (P_2, a_\bullet^2), (Q_2, b_\bullet))$ meets $G_d^r(Z_2, (P_2, \bar{a}_\bullet^2), (Q_2, b_\bullet))$, so we conclude that every connected component of the part $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))^{b_\bullet^1, a_\bullet^2}$ meets the part $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))^{\bar{b}_\bullet^1, \bar{a}_\bullet^2}$. But now we have reduced to the previously handled case, so every connected component of the part $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))^{\bar{b}_\bullet^1, \bar{a}_\bullet^2}$ meets the subset $G_d^r(Z_1, (P_1, \bar{a}_\bullet), (Q_1, \bar{b}_\bullet^1)) \times G_d^r(Z_2, (P_2, \bar{a}_\bullet^2), (Q_2, b_\bullet))$, proving the desired statement for X_0 . The desired statement for $G_d^r(X_\eta, (P_\eta, a_\bullet), (Q_\eta, b_\bullet))$ then follows from Lemma 4.1, and considering a versal family of curves containing X_η and consisting entirely of curves X and P, Q such that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is geometrically reduced of dimension ρ , the same lemma implies the desired statement for a general 2-marked curve.

We next show that when X is a general 2-marked curve, and $\hat{\rho} \geq 1$, the space $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is connected. The proof will again be by induction on genus, with the base cases $g = 0, 1$ having been proved in Proposition 2.1. For the induction step, we consider the same family $\pi : X \rightarrow B$ as before. Using the aforementioned constancy of geometric connected components (again, both in the family π , and in a suitable versal family), we see that it is enough to show that $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ is connected.

Now, for any vanishing sequences b_\bullet^1 and a_\bullet^2 , we have by Proposition 2.1 that $G_d^r(Z_1, (P_1, a_\bullet), (Q_1, b_\bullet^1))$ is connected when it is nonempty, and by the induction hypothesis, $G_d^r(Z_2, (P_2, a_\bullet^2), (Q_2, b_\bullet))$ is connected when $\hat{\rho}_2 > 0$. Moreover, by the statement we have just proved, even when $\hat{\rho}_2 = 0$, we have that if we increase one term of a_\bullet^2 while maintaining the condition that $\hat{\rho}_2 = 0$, every connected component $G_d^r(Z_2, (P_2, a_\bullet^2), (Q_2, b_\bullet))$ will meet the resulting closed subscheme. We will consider complementary pairs of b_\bullet^1 and a_\bullet^2 with $\hat{\rho}_1 \geq 0$ and $\hat{\rho}_2 \geq 0$ and show that the parts $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))^{b_\bullet^1, a_\bullet^2}$ corresponding to any two such pairs can be connected to one another through a sequence of moves which increase exactly one entry of b_\bullet^1 or a_\bullet^2 at a time.

There are two types of moves we consider. For the first, we assume that $\rho_1 > 0$. Let j_0 be minimal such that $a_{j_0} + b_{r-j_0}^1$ attains its minimum value, which is necessarily at most $d - 1$. By the minimality assumptions, either $j_0 = 0$ or $b_{r-j_0+1}^1 \geq b_{r-j_0}^1 + 2$; in either case, we can replace $b_{r-j_0}^1$ with $b_{r-j_0}^1 + 1$ to obtain a valid sequence which still has $\hat{\rho}_1 \geq 0$, but with ρ_1 decreased by 1. We can then replace $a_{j_0}^2$ with $a_{j_0}^2 - 1$, and we obtain a new pair of complementary sequences which still satisfy the nonemptiness condition.

For the second type of move, suppose that $\hat{\rho}_2 = 0$, and that if also $\hat{\rho}_1 = 0$, then the j with $a_j + b_{r-j}^1 = d$ satisfies $a_j^2 + b_{r-j} \geq d - (g - 1)$. Then the second type of move is, for a certain j_0 , to replace $a_{j_0}^2$ by $a_{j_0}^2 - 1$, and $b_{r-j_0}^1$ by $b_{r-j_0}^1 + 1$. This will result in a case with $\hat{\rho}_1 \geq 0$ still, but $\hat{\rho}_2 > 0$. The desired j_0 must therefore satisfy the following conditions: $a_{j_0} + b_{r-j_0}^1 \leq d - 1$; $a_{j_0}^2 + b_{r-j_0} > d - (g - 1)$; and either $j_0 = 0$ or $a_{j_0-1}^2 < a_{j_0}^2 - 1$. Moreover, if $\hat{\rho}_1 = 0$, we need $a_{j_0} + b_{r-j_0}^1 < d - 1$. Now, if $\hat{\rho}_1 > 0$, the condition $a_{j_0} + b_{r-j_0}^1 \leq d - 1$ is automatic, and we obtain such

a j_0 by letting j_0 be minimal with $a_{j_0}^2 + b_{r-j_0}$ maximal. If $\widehat{\rho}_1 = 0$, because $\widehat{\rho} > 0$ Lemma 4.2 implies that we must have some j with $a_j^2 + b_{r-j} > d - (g - 1)$, and $a_j + b_{r-j}^1 < d - 1$. Let j_0 be minimal such that $a_{j_0}^2 + b_{r-j_0}$ attains the maximal value subject to the condition that $a_{j_0} + b_{r-j_0}^1 < d - 1$. By construction, if $j_0 > 0$, we either have $a_{j_0-1}^2 + b_{r-j_0+1} < a_{j_0}^2 + b_{r-j_0}$ or $a_{j_0-1} + b_{r-j_0+1}^1 > a_{j_0} + b_{r-j_0}^1$ and in either case we see that $a_{j_0-1}^2 < a_{j_0}^2 - 1$, as claimed. Thus, we have produced j_0 satisfying the desired conditions.

Now, suppose we start with $\widehat{\rho}_1 > 0$; applying the first type of move repeatedly necessarily brings us eventually to the single case that $a_j + b_{r-j}^1 = d - 1$ for all j . Then applying the second type of move once, we see that this case is connected to a case with $\widehat{\rho}_2 > 0$. Finally, we show that any case with $\widehat{\rho}_1 = 0$ is connected to a case with $\widehat{\rho}_1 > 0$. Let j_0 be the value with $a_{j_0} + b_{r-j_0}^1 = d$; then we must have either $j_0 = 0$ or $a_{j_0-1} + b_{r-j_0+1}^1 = d - 1$, so in either case we can decrease $b_{r-j_0}^1$ by 1. If $a_{j_0}^2 + b_{r-j_0} < d - (g - 1)$, we can increase $a_{j_0}^2$ by 1, and will still necessarily have $\widehat{\rho}_2 \geq 0$, connecting us to the desired case with $\widehat{\rho}_1 > 0$. On the other hand, if $a_{j_0}^2 + b_{r-j_0} \geq d - (g - 1)$, then either $\widehat{\rho}_2 > 0$, or we can apply the second type of move to connect to a case with $\widehat{\rho}_2 > 0$. Once $\widehat{\rho}_2 > 0$, we again find that we can increase $a_{j_0}^2$ by 1, yielding the desired case with $\widehat{\rho}_1 > 0$. To summarize, we have seen first that all cases are connected to one another, and second, that at least some cases have $\widehat{\rho}_2 > 0$, so that the corresponding product space is connected. Thus, we have represented $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ as a union of the parts $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))^{b_\bullet^1, a_\bullet^2}$, where each part may be represented as the vertex of a connected graph, and the edges correspond to the above-described moves. Using what we have already proved for the case $\widehat{\rho}_2 = 0$, we see that the part corresponding to each vertex has every connected component meeting the part corresponding to any adjacent vertex, and we also have that at least one of the parts is connected. It follows that the union is connected, as desired. \square

We now easily conclude our main theorem.

Proof of Theorem 1.2. We have already proved the first statement, as the $n = 1$ case of Corollary 3.5. Theorem 4.3 gives us that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ is connected when X is a general 2-marked curve and $\widehat{\rho} \geq 1$. Finally, because the moduli spaces of linear series are universally open over the locus on \mathcal{M}_g on which they have the expected dimension (as explained for instance in the proof of Theorem 3.4), we can conclude the connectedness assertion of Theorem 1.2 by specialization from general curves, using Corollary 15.5.4 of [GD66]. \square

Example 4.4. We see that the connectedness statement fails quite quickly when we have $\widehat{\rho} = 0$. Indeed, consider the case $g = 2$, $r = 2$, $d = 6$, with $a_\bullet = 0, 2, 3$ and $b_\bullet = 0, 3, 5$. This has $\rho = 1$ but $\widehat{\rho} = 0$. Consider a degeneration to two genus-1 components. We see that the only possibilities for b_\bullet^1 which have both $\widehat{\rho}_1 \geq 0$ and $\widehat{\rho}_2 \geq 0$ are $1, 3, 6$; $2, 3, 6$; $1, 4, 5$; or $2, 4, 5$. The first pair and second pair correspond to intersecting components of the limit linear series space, but neither component from the first pair intersects either component of the second pair, so we see that the limit linear series space is disconnected in this case. Since the space is nonetheless reduced of the expected dimension, it follows that $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ likewise has two connected components for (X, P, Q) a general 2-marked curve of genus 2.

We also consider the second part of Theorem 4.3 in this case. The only possible choice of \bar{a}_\bullet will be 0, 2, 4. This results in two limit linear series, with b_\bullet^1 given by 1, 3, 6 or 1, 4, 5. Thus, we see that only two of the four irreducible components of $G_d^r(X_0, (P_1, a_\bullet), (Q_2, b_\bullet))$ meet $G_d^r(X_0, (P_1, \bar{a}_\bullet), (Q_2, b_\bullet))$, but that, as the theorem asserts, both connected components meet it.

Finally, we note that we can conclude connectedness more generally for spaces of limit linear series:

Corollary 4.5. *In Situation 3.1, given also vanishing sequences a_\bullet, b_\bullet , suppose also that $\hat{\rho} \geq 1$, and that the space $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ is reduced of dimension ρ . Then $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ is connected.*

Note that by Corollary 3.5, the dimension and reducedness hypotheses are satisfied in particular when the components of X_0 are general as 2-marked curves.

Proof. Knowing connectedness for a general smooth curve, the desired statement follows by specialization just as in the proof of the last part of Theorem 1.2. \square

Remark 4.6. Even when $\hat{\rho} = 0$, our techniques reduce the problem of finding the number of connected components of $G_d^r(X, (P, a_\bullet), (Q, b_\bullet))$ for a general 2-marked curve to a purely combinatorial question. Indeed, if we use Theorem 3.4, Corollary 3.3, and Proposition 15.5.7 of [GD66], we see that the answer is determined by the number of connected components of $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$, where X_0 is a chain of curves of genus 1 with suitably general marked points. As is well known (and follows for instance from Proposition 2.1), the irreducible components of $G_d^r(X_0, (P_1, a_\bullet), (Q_n, b_\bullet))$ correspond to choices of tuples of complementary vanishing sequences at the nodes which have $\hat{\rho}_i \geq 0$ for all i , and we can likewise determine which pairs of components have nonempty intersection by looking at the vanishing sequences involved, so the question becomes entirely combinatorial.

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